Motion Generation and Control for the Chaplygin Beanie

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Abstract

In this paper the Chaplygin beanie, a modified Chaplygin sleigh, is introduced. Motion generation and control results are established. In particular, it is established that both the beanie's heading angle and speed can be controlled asymptotically with a single input. The surprising control of two outputs with a single input results from the exploitation of a nonholonomic constraint.

1 Introduction

The Chaplygin sleigh is a rigid planar body with a blade-like wheel located at the sleigh's center of mass. The sleigh can pivot about its wheel and move parallel to its heading, but it cannot slide in a direction perpendicular to its heading. The wheel provides a *nonholonomic constraint*, which makes the analysis interesting and provides a "hidden" control input. The Chaplygin sleigh has been studied extensively. See the recent papers [OZ] and [BMZ] for background and further references.

We introduce the *Chaplygin beanie*,[1] a modified Chaplygin sleigh, described as follows (see Figure 1). The wheel is offset towards the rear of the cart by distance a, and an actuated disk that spins about its center is positioned at the center of mass (x, y) of the sleigh. Motion generation and control of the beanie's heading and speed are achieved by actuating the disk.

The control objective is the following: given initial conditions, find a control law for actuating the disk to achieve a desired heading and speed. We propose such a control law and prove that it achieves the control objective, subject to a certain restriction.

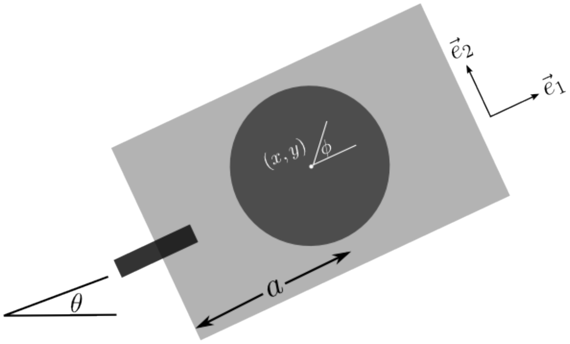

Figure 1: The Chaplygin beanie

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[1] The name Chaplygin beanie is a combination of "Chaplygin sleigh" and "Elroy's beanie," two famous examples in nonholonomic mechanics and geometric control.

2 Equations of Motion

In this section we construct the nonholonomic momenta for the Chaplygin beanie and derive the corresponding momentum equations. We follow the approach outlined in §4 of the seminal [BKMM] paper.

We equip our system with the structure of a trivial principal fiber bundle. The position and orientation of the sleigh is represented by an element of the Lie group $G = SE(2)$ of affine transformations on $\mathbb{R}^2$. The shape space is $F = SO(2)$, describing the angle of the disk relative to a reference heading pointing along the body-fixed $\vec{e}_1$-axis. The bundle is $Q = G \times F$, equipped with the canonical projection $\pi : Q \to F : (x, y, \theta, \phi) \mapsto \phi$ and the group action $\Phi : G \times Q \to Q$ given by

$$\Phi(a^1, a^2, \psi, x, y, \theta, \phi) = (a^1 - x\cos\psi + y\sin\psi, a^2 - x\sin\psi + y\cos\psi, \theta + \psi, \phi).$$

Let M denote the combined mass of the sleigh and disk, C be the moment of inertia of the sleigh about its wheel, and B the moment of inertia of the disk (or beanie) about its center. From Figure 1, we read off the Lagrangian $L : TQ \to \mathbb{R}$ as

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}C\dot{\theta}^2 + \frac{1}{2}B(\dot{\theta} + \dot{\phi})^2. \tag{1}$$

The conjugate momenta $p_x, p_y, p_\theta, p_\phi$ are the components of the fiber derivative $\mathbb{F}L : TQ \to T^*Q$ computed in local coordinates. The Legendre transform $(\mathbf{q}, \dot{\mathbf{q}}) \mapsto (\mathbf{q}, \frac{\partial L}{\partial \dot{\mathbf{q}}})$ gives them as

$$p_x := \frac{\partial L}{\partial \dot{x}} = M\dot{x}, \tag{2}$$

$$p_y := \frac{\partial L}{\partial \dot{y}} = M\dot{y}, \tag{3}$$

$$p_\theta := \frac{dL}{d\dot{\theta}} = (C + B)\dot{\theta} + B\dot{\phi} \tag{4}$$

$$p_\phi := \frac{dL}{d\dot{\phi}} = B(\dot{\theta} + \dot{\phi}). \tag{5}$$

The sleigh is constrained in that it cannot translate perpendicular to the blade, i.e. the $\vec{e}_2$ direction. One sees from Figure 1 that, in fact, we must have

$$a\dot{\theta} = (\dot{x}, \dot{y}) \cdot \vec{e}_2 = (\dot{x}, \dot{y}) \cdot (-\sin\theta, \cos\theta) = -\dot{x}\sin\theta + \dot{y}\cos\theta.$$

In other words, our system is subjected to the constraint equation

$$\dot{x}\sin\theta - \dot{y}\cos\theta + a\dot{\theta} = 0. \tag{6}$$

At any $q = (x, y, \theta, \phi) \in Q$, define the constraint one-form $\omega_q \in T_q^*Q$ as

$$\omega_q = \sin\theta\, dx|_q - \cos\theta\, dy|_q + a\, d\theta|_q.$$

Eq. (6) says that ω_q annihilates velocity vectors $\dot{x}\frac{\partial}{\partial x}|_q + \dot{y}\frac{\partial}{\partial y}|_q + \dot{\theta}\frac{\partial}{\partial \theta}|_q + \dot{\phi}\frac{\partial}{\partial \phi}|_q \in T_qQ$ that are tangent to the sleigh's trajectory at q. Note that, since $\omega_q \neq df$ for any function $f(x, y, \theta, \phi)$ of the sleigh's coordinates, the constraint is nonintegrable, i.e. nonholonomic. At each point $q \in Q$, the constraint equation, Eq. (6), determines a subspace $\mathcal{D}_q \subset T_qQ$ consisting of precisely those velocity vectors in T_qQ satisfying the constraint equation. A velocity vector $\dot{q} \in T_qQ$ satisfies the constraints iff $\dot{q} \in D_q$. A trajectory $(q(t), \dot{q}(t))$ satisfies the constraints iff $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t.

At any $q = (x, y, \theta, \phi) \in Q$, a straightforward calculation [2] shows that

$$\mathcal{D}_q = \operatorname{span}\left\{-a\sin\theta\frac{\partial}{\partial x} + a\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}, \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \frac{\partial}{\partial\phi}\right\}. \tag{7}$$

Of course the choice of vectors that span $\mathcal{D}_q$ is not unique, but it must be noted that the choice of basis for $\mathcal{D}_q$ determines the physical meaning of the resulting nonholonomic momenta. Accordingly, the physical nature of the problem and the desired control strategy must be taken into account when choosing a basis for the constraint distribution. The choice made in Eq. (7) was motived by our desire to control the sleigh's heading – which amounts to requiring the angular momentum to approach zero. Indeed, we will see that the choice made in Eq. (7) leads to a linear momentum along the $\vec{e}_1$-axis and an angular momentum about the cart's pivot point; this is the natural choice for control objective we are considering.

The tangent space to the group orbit through $q \in Q$ is

$$T_q(\mathrm{Orb}(q)) = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}\right\}. \tag{8}$$

At each $q \in Q$, we define $\mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))$. From Eqs. (7) and (8),

$$\mathcal{S}_q = \operatorname{span}\left\{-a\sin\theta\frac{\partial}{\partial x} + a\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta}, \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right\}. \tag{9}$$

The vector fields ξ_Q and η_Q defined by

$$\xi_Q = -a\sin\theta\frac{\partial}{\partial x} + a\cos\theta\frac{\partial}{\partial y} + \frac{\partial}{\partial\theta} \tag{10}$$

$$\eta_Q = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \tag{11}$$

are such that $\mathcal{S}_q = \operatorname{span}\{\xi_Q(q), \eta_Q(q)\}$ at each point $q \in Q$. Next, for each $q \in Q$, define $\mathfrak{g}^q = \{\xi \in \mathfrak{g} : \xi_Q(q) \in \mathcal{S}_q\}$. The corresponding bundle over Q whose fiber at q is given by $\mathfrak{g}^q$ is denoted by $\mathfrak{g}^{\mathcal{D}}$. The nonholonomic momentum map is the bundle map $J^{\mathrm{nhc}} : TQ \to (\mathfrak{g}^{\mathcal{D}})^*$ defined by $\langle J^{\mathrm{nhc}}(v_q), \xi\rangle = \langle \mathbb{F}L(v_q), \xi\rangle$, where $\xi \in \mathfrak{g}^q$. In coordinates,

$$\langle J^{\mathrm{nhc}}(v_q), \xi\rangle = \frac{\partial L}{\partial \dot{q}^i}(\xi_Q)^i. \tag{12}$$

See §4 of [BKMM] for these definitions and details.

From Eqs. (2)-(5) and (10)-(12), the nonholonomic momenta are

$$J_{LT} := J^{\mathrm{nhc}}_{\xi} = M\dot{x}\cos\theta + M\dot{y}\sin\theta \tag{13}$$

$$J_{RW} := J^{\mathrm{nhc}}_{\eta} = (Ma^2 + C + B)\dot{\theta} + B\dot{\phi} \tag{14}$$

where we have aptly denoted $J_{LT} = J^{\mathrm{nhc}}_{\xi}$ and $J_{RW} = J^{\mathrm{nhc}}_{\eta}$ since, physically, J_{LT} represents the linear momentum of longitudinal translation along the sleigh's heading and J_{RW} represents the rotational angular momentum of the sleigh and spinning disk about the wheel. Note from Eq. (14) that

$$\dot{\theta} = \frac{J_{RW} - B\dot{\phi}}{Ma^2 + C + B}. \tag{15}$$

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[2] To arrive at this in practice, one may find a basis for the null space of the matrix associated to the constraint equation(s). One can then rotate, scale, or otherwise modify the resulting basis so that it make physical sense. $\mathcal{D}_q$ is the span of the resulting basis.

Section 4 of [BKMM] gives the following momentum equations in local coordinates:

$$\dot{J}_{LT} = \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} \xi \right]_Q^i , \tag{16}$$

$$\dot{J}_{RW} = \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} \eta \right]_Q^i , \tag{17}$$

where $\xi, \eta \in \mathfrak{g}^q$ are the Lie algebra elements whose infinitesimal generators give the vector fields ξ_Q and η_Q that span S_q. The notation $[\frac{d}{dt}\xi]_Q^i$ signifies the ith component of the infinitesimal generator vector field generated by $\frac{d}{dt}\xi$, and similarly for $[\frac{d}{dt}\eta]_Q^i$. A side calculation shows that

$$\begin{aligned} \xi &= (\cos\theta, \sin\theta, 0)^T, \\ \eta &= (y - a\sin\theta, -x + a\cos\theta, 1)^T \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\xi &= (-\dot{\theta}\sin\theta, \dot{\theta}\cos\theta, 0)^T, \\ \frac{d}{dt}\eta &= (\dot{y} - a\dot{\theta}\cos\theta, -\dot{x} - a\dot{\theta}\sin\theta, 0)^T, \end{aligned}$$

where we are using the canonical identification between $\mathfrak{se}(2)$ and column vectors in $\mathbb{R}^3$.

Computing the right-hand sides of Eqs. (16) and (17) gives the two momentum equations

$$\begin{aligned} \dot{J}_{LT} &= M\dot{x}(-\dot{\theta}\sin\theta) + M\dot{y}(\dot{\theta}\cos\theta) = Ma\dot{\theta}^2 \\ &= \frac{Ma}{(Ma^2 + C + B)^2}(J_{RW} - B\dot{\phi})^2, && (18) \\ \dot{J}_{RW} &= M\dot{x}(\dot{y} - a\dot{\theta}\cos\theta) + M\dot{y}(-\dot{x} - a\dot{\theta}\sin\theta) \\ &= \frac{-a}{Ma^2 + C + B}(J_{RW} - B\dot{\phi})J_{LT}, && (19) \end{aligned}$$

which, together with Eq. (15) and a control law (postulated in the next section) form the equations of motion for our system.

3 Control

Let us restate the control objective:

> **Control Objective**: Given initial conditions for the Chaplygin beanie, determine how to actuate the disk in order to achieve straight-line motion along a desired heading and with a desired speed.

Without loss of generality, we may assume the desired heading is $\theta = 0$. On the basis of physical reasoning, we see that if the rotating disk is given a positive acceleration, i.e. $\ddot{\phi} > 0$, then the sleigh will tend to rotate clockwise. Conversely, if $\ddot{\phi} < 0$ the sleigh will rotate counterclockwise. By analogy with a mass-spring system, we suggest the control law

$$\ddot{\phi} = k\theta \tag{20}$$

for some unknown constant k. That is, we apply a torque to the disk proportional in magnitude to the error in the desired heading. The physical considerations just discussed also suggest that $k > 0$.

We call the constant k the *gain* of the controller. In fact, we will see that if k is any positive value then the desired heading will be achieved, and that any speed at least as large as the initial speed can be achieved by tuning the value of k. For k negative, the cart's motion becomes unbounded.

Our first step in the analysis of the control law is to write down the complete system of equations, namely Eqs. (18), (19), (15), and (20) in the form of a first-order system of ODEs. Doing so, we obtain

$$\begin{aligned}
\dot{J}_{LT} &= \frac{Ma}{(Ma^2+C+B)^2}(J_{RW}-B\alpha)^2, \\
\dot{J}_{RW} &= \frac{-a}{Ma^2+C+B}(J_{RW}-B\alpha)J_{LT}, \\
\dot{\theta} &= \frac{J_{RW}-B\alpha}{Ma^2+C+B}, \\
\dot{\phi} &= \alpha, \\
\dot{\alpha} &= k\theta.
\end{aligned} \tag{21}$$

Let us simplify this system as follows. Let $C_1 = \frac{Ma}{(Ma^2+C+B)^2}$, $C_2 = \frac{-a}{Ma^2+C+B}$, $C_3 = \frac{1}{Ma^2+C+B}$, and note that $C_1 > 0$, $C_2 < 0$, and $C_3 > 0$. The change of variables $u = J_{LT}$, $v = J_{RW} - B\alpha$, and $w = \theta$ transforms (21) to

$$\begin{aligned}
\dot{u} &= C_1 v^2, \\
\dot{v} &= C_2 uv - Bkw, \\
\dot{w} &= C_3 v,
\end{aligned} \tag{22}$$

together with the reconstruction equations

$$\begin{aligned}
\dot{\phi} &= \alpha, \\
\dot{\alpha} &= kw.
\end{aligned}$$

We make some immediate observations:

1. According to the control objective, we are interested in the quantities u (i.e. J_{LT}, the linear momentum) and w (i.e. θ, the heading). Since the u, v, w equations are decoupled from the ϕ, α equations, we will restrict attention to the first three equations in system (22). If desired, $\phi(t)$ and $\alpha(t)$ can be reconstructed by simple integration after solving (22).
2. Note that $\dot{u} \geq 0$. Thus the linear momentum (and so the linear speed) of the cart is always increasing in magnitude. What's more, if $u = v = 0$ but $w \neq 0$ initially, then after a small Δt we have $u > 0$. That is, the controller generates forward motion of the cart, from rest, if its initial heading is not the desired one.
3. The set of fixed points of (22) is exactly the set where $v = w = 0$, i.e. the u-axis. This corresponds to motion along a straight line in the desired heading.
4. The constants C_1, C_2, C_3, B depend on the physical construction of the sleigh; the only parameter we may tune is the gain k.

Assume $k \neq 0$. We begin by searching for a first integral of system (22), i.e. a scalar-valued function $G : \mathbb{R}^3 \to \mathbb{R}$ such that $\dot{G}(u, v, w) = \nabla G \cdot f = 0$. This condition reads

$$\left(\frac{\partial G}{\partial u}\right)(C_1 v^2) + \left(\frac{\partial G}{\partial v}\right)(C_2 uv - Bkw) + \left(\frac{\partial G}{\partial w}\right)(C_3 v) = 0.$$

By inspection, a good candidate is $G(u,v,w) = \frac{1}{2}\Gamma_1 u^2 + \frac{1}{2}\Gamma_2 v^2 + \frac{1}{2}\Gamma_3 w^2$ for some unknown constants $\Gamma_1, \Gamma_2, \Gamma_3$. With this G, the condition $\nabla G \cdot f = 0$ becomes

$$(\Gamma_1 C_1 + \Gamma_2 C_2) u v^2 + (\Gamma_3 C_3 - \Gamma_2 B k) v w = 0.$$

Thus if $\Gamma_1 C_1 + \Gamma_2 C_2 = 0$ and $\Gamma_3 C_3 - \Gamma_2 B k = 0$, then G will be a first integral of (22). Indeed, if we let $\Gamma_1 = 1/M$, so that G has units of energy, then $\Gamma_2 = -C_1/MC_2 = \frac{1}{Ma^2+C+B}$, and $\Gamma_3 = \Gamma_2 Bk/C_3 = Bk$. Thus

$$G(u,v,w) = \frac{1}{2M}u^2 + \frac{1}{2(Ma^2+C+B)}v^2 + \frac{Bk}{2}w^2$$

is a first integral for (22). We record this observation as follows:

Proposition 1. *For $k \neq 0$, the energy function $G : \mathbb{R}^3 \to \mathbb{R} : (u,v,w) \mapsto \frac{1}{2}\Gamma_1 u^2 + \frac{1}{2}\Gamma_2 v^2 + \frac{1}{2}\Gamma_3 w^2$ with $\Gamma_1 = \frac{1}{M}$, $\Gamma_2 = \frac{1}{Ma^2+C+B}$, and $\Gamma_3 = Bk$ is a first integral for system (22).*

The level sets of G are invariant sets for the flow of (22). Note that if $k > 0$, then $\Gamma_i > 0$ for all i, and the level sets of G are ellipsoids centered at the origin. If $k < 0$, the level sets are hyperboloids, and each of $u(t), v(t), w(t)$ becomes unbounded as $t \to \infty$. See Figures 2,3 below. (The mesh lines in those figures do not represent trajectories.) Thus we require the gain k to be positive.

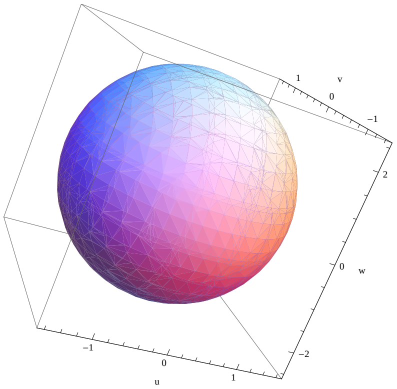

Figure 2: Energy level surface with $k > 0$

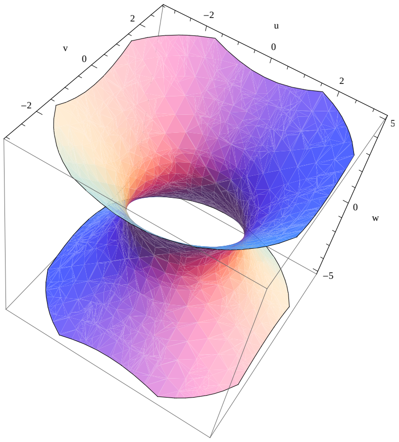

Figure 3: Energy level surface with $k < 0$

We now recall a general result in the theory of dynamical systems that will be useful to the study of our system (see Exercise 1.172 in [C]).

Theorem 2. *Suppose that the differential equation $\dot{x} = f(x)$ with flow ϕ_t has a compact invariant set K, and $V : K \to \mathbb{R}$ is a C^1 function such that $\dot{V}(x) \leq 0$ for all $x \in K$. Let Ω denote the largest invariant set in $\{x \in K : \dot{V}(x) = 0\}$. Then every solution in K approaches Ω as $t \to \infty$.*

Using the preceding theorem, we prove the following result about system (22).

Theorem 3. *Fix $k > 0$ and let ϕ_t be the flow of system (22). If $x_0 = (u_0, v_0, w_0)^T \in \mathbb{R}^3$ is not a fixed point of (22), then $\lim_{t\to\infty} \phi_t(x_0) = (u_*, 0, 0)$, where*

$$u_* = \sqrt{u_0^2 + (\Gamma_2/\Gamma_1)v_0^2 + (\Gamma_3/\Gamma_1)w_0^2} = \sqrt{u_0^2 + \left(\frac{M}{Ma^2+C+B}\right)v_0^2 + (BkM)w_0^2}.$$

Proof. The initial condition $x_0 = (u_0, v_0, w_0)^T \in \mathbb{R}^3$ belongs to the ellipsoid G_{x_0} defined by $G_{x_0} = \{(u, v, w)^T \in \mathbb{R}^3 : G(u, v, w) = G(u_0, v_0, w_0)\}$. Note that G_{x_0} is compact, and it is an invariant set for the flow ϕ_t by Proposition 1. Define $\mathcal{H} : G_{x_0} \to \mathbb{R}$ by $\mathcal{H}(u, v, w) = -u$. Then $\dot{\mathcal{H}} = -\dot{u} = -C_1 v^2 \leq 0$. Let Ω be the largest invariant set in $\mathcal{E} = \{x \in G_{x_0} : \dot{\mathcal{H}} = 0\} = (G_{x_0} \cap \{v = 0\})$. Note that $\mathcal{E}$ is the elliptical cross-section of G_{x_0} lying in the (u, w)-plane. We now determine the set Ω.

If $p = (u, 0, w)^T \in \mathcal{E}$ has $w \neq 0$, then $(\dot{u}, \dot{v}, \dot{w})|_p = (0, -Bkw, 0)^T \neq 0$ is transverse to $\mathcal{E}$. Consequently, $p \notin \Omega$. It follows that $\Omega \subset \mathcal{E} \cap \{w = 0\} = \{\pm(u_*, 0, 0)\}$, where

$$u_* = \sqrt{u_0^2 + (\Gamma_2/\Gamma_1)v_0^2 + (\Gamma_3/\Gamma_1)w_0^2}$$

is determined from $G(u_*, 0, 0) = G(u_0, v_0, w_0)$. Each of $\pm(u_*, 0, 0)$ is a fixed point and belongs to Ω, so in fact $\Omega = \{\pm(u_*, 0, 0)\}$.

It now follows from Theorem 2 that $\lim_{t\to\infty} \phi_t(x_0) = (u_*, 0, 0)$ or $\lim_{t\to\infty} \phi_t(x_0) = (-u_*, 0, 0)$. Since $x_0 = (u_0, v_0, w_0)$ is not a fixed point, we have $u_0 > -u_*$. Moreover, $\dot{u} = C_1 v^2 \geq 0$ implies $u(t) \geq u_0 > -u_*$ for all time. Thus $\lim_{t\to\infty} \phi_t(x_0) = (-u_*, 0, 0)$ is impossible. Consequently, $\lim_{t\to\infty} \phi_t(x_0) = (u_*, 0, 0)$, as required. ■

4 Conclusion

In this paper we introduced the *Chaplygin beanie* and derived its equations of motion. We proposed a physically realistic control law for actuating the disk to achieve a desired heading and speed, and we proved that it works. In future work we will examine additional control laws for the Chaplygin beanie and study this system in relation to other simple nonholonomic mechanical systems.

5 References

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