

Notation: The symbol \sqcup denotes a disjoint union.

Let X be a nonempty set equipped with a σ -algebra \mathcal{M} . A **measure** on \mathcal{M} is a set function $\mu : \mathcal{M} \rightarrow [0, \infty]$ such that

- (i) $\mu(\emptyset) = 0$,
- (ii) $\mu(\bigsqcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$ for any disjoint collection $\{E_j\}_1^\infty \subset \mathcal{M}$.

One also says μ is a measure on (X, \mathcal{M}) or simply a measure on X if \mathcal{M} is understood. Property (ii) is called **countable additivity**. If μ satisfies (i) and

(ii') If $\{E_j\}_1^n \subset \mathcal{M}$ is a sequence of disjoint sets then $\mu(\bigcup_1^n E_j) = \sum_1^n \mu(E_j)$, but not necessarily (ii), then μ is called a **finitely additive measure** on \mathcal{M} . A measure μ is called **complete** if $\mu(E) = 0$ implies $F \in \mathcal{M}$ for all $F \subset E$.

A **measurable space** is a pair (X, \mathcal{M}) where X is a nonempty set and \mathcal{M} is a σ -algebra on X . The members of \mathcal{M} are called **measurable sets**. If $E \in \mathcal{M}$ has $\mu(E) = 0$, then it is called a set of **measure zero**. If a statement is true for all $x \in X$ except possibly for points in a set of measure zero, then the statement is true **almost everywhere** (abbreviated a.e.), or for **almost every** x .

If μ is a measure on (X, \mathcal{M}) , the triple (X, \mathcal{M}, μ) is called a **measure space**. If $\mu(X) < \infty$ then μ is called **finite**. If $E = \bigcup_1^\infty E_n$ where $\{E_n\}_1^\infty \subset \mathcal{M}$ and $\mu(E_n) < \infty$ for all n , then E is called **σ -finite** for μ . If X is σ -finite for μ , then μ is called **σ -finite**. A σ -finite measure is “almost” finite, in that many theorems can be extended from the finite to the σ -finite case by decomposing X into a countable union of sets of μ -finite measure and applying a $\frac{1}{2^n}$ argument.

The following theorem establishes basic properties of measures. But first, recall the definition of \liminf and \limsup of a sequence of sets $\{E_n\}_1^\infty$, namely

$$\liminf E_n = \bigcup_{n=1}^{\infty} \bigcap_{j \geq n} E_j,$$

$$\limsup E_n = \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} E_j$$

Theorem 1 (Properties of Measures, I). *Let (X, \mathcal{M}, μ) be a measure space.*

- (a) $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.
- (b) $\mu(E \cap F) < \infty$ implies $\mu(E \setminus F) = \mu(E) - \mu(E \cap F)$.
- (c) $F \subset E$ and $\mu(F) < \infty$ imply $\mu(E \setminus F) = \mu(E) - \mu(F)$.
- (d) $E \subset F$ implies $\mu(E) \leq \mu(F)$.
- (e) $\mu(\bigcup_1^\infty E_n) \leq \sum_{n=1}^\infty \mu(E_n)$.
- (f) A countable union of sets of measure zero has measure zero.
- (g) $E_1 \subset E_2 \subset \dots$ implies $\mu(\bigcup_1^\infty E_n) = \lim \mu(E_n)$.
- (h) $\mu(\bigcup_1^\infty E_j) = \lim \mu(\bigcup_1^n E_j)$.
- (i) $E_1 \supset E_2 \supset \dots$ and some $\mu(E_N) < \infty$ imply $\mu(\bigcap_1^\infty E_n) = \lim \mu(E_n)$.
- (j) Some $\mu(E_N) < \infty$ implies $\mu(\bigcap_1^\infty E_n) = \lim \mu(\bigcap_1^n E_j)$.
- (k) $\mu(\liminf E_n) \leq \liminf \mu(E_n)$.
- (l) $\mu(\limsup E_n) \geq \limsup \mu(E_n)$, provided $\mu(\bigcup_N^\infty E_n) < \infty$ for some N .
- (m) For $E \in \mathcal{M}$, the function $\mu_E : \mathcal{M} \rightarrow [0, \infty] : A \mapsto \mu(A \cap E)$ is a measure.
- (n) A linear combination (with coefficients ≥ 0) of measures is a measure.

Property (d) is referred to as **monotonicity**, (e) as **subadditivity**, (g) and (h) as **continuity from below**, and (i) and (j) as **continuity from above**.

Proof. (a) Since $E \cup F = (E \setminus F) \sqcup (E \cap F) \sqcup (F \setminus E)$, it follows that

$$\begin{aligned} \mu(E \cup F) + \mu(E \cap F) &= \mu(E \setminus F) + \mu(E \cap F) + \mu(F \setminus E) + \mu(E \cap F) \\ &= \mu(E) + \mu(F), \end{aligned}$$

the last equality being true because $E = (E \setminus F) \sqcup (E \cap F)$ and similarly for F .

(b) Write $E = (E \setminus F) \sqcup (E \cap F)$. Then $\mu(E) = \mu(E \setminus F) + \mu(E \cap F)$. Since $\mu(E \cap F) < \infty$, subtracting it from both sides gives $\mu(E \setminus F) = \mu(E) - \mu(E \cap F)$.

(c) follows from (b).

(d) Write $F = E \sqcup (F \setminus E)$. Then $\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$.

(e) Put $F_n = E_n \setminus \bigcup_1^{n-1} E_j$. By (a), $\mu(F_n) \leq \mu(E_n)$. Also, $\bigcup_1^\infty E_n = \bigsqcup_1^\infty F_n$, so $\mu(\bigcup_1^\infty E_n) = \mu(\bigsqcup_1^\infty F_n) = \sum_1^\infty \mu(F_n) \leq \sum_1^\infty \mu(E_n)$.

(f) follows from (e).

(g) Put $E_0 = \emptyset$ and $F_n = E_n \setminus E_{n-1}$. Then $\bigsqcup_1^\infty F_n = \bigcup_1^\infty E_n$ and $\bigsqcup_1^n F_j = E_n$. Therefore, $\mu(E_n) = \mu(\bigsqcup_1^n F_j) = \sum_1^n \mu(F_j)$. Consequently, $\mu(\bigcup_1^\infty E_n) = \mu(\bigsqcup_1^\infty F_n) = \sum_1^\infty \mu(F_n) = \lim \sum_1^n \mu(F_j) = \lim \mu(E_n)$.

(h) follows from (g) after defining $E_n = \bigcup_1^n E_j$.

(i) Put $F_n = E_N \setminus E_n$. Then $F_1 \subset F_2 \subset \dots$, so (f) implies $\mu(\bigcup_1^\infty F_n) = \lim \mu(F_n)$. Now, $\bigcup_1^\infty F_n = \bigcup_1^\infty E_N \setminus E_n = E_N \setminus \bigcap_1^\infty E_n$, and since $E_N \cap (\bigcap_1^\infty E_n) = \bigcap_1^\infty E_n \subset E_N$ has finite measure, it follows from (c) that $\mu(\bigcup_1^\infty F_n) = \mu(E_N \setminus \bigcap_1^\infty E_n) = \mu(E_N) - \mu(\bigcap_1^\infty E_n)$. On the other hand, for $n \geq N$, $E_N \cap E_n = E_n \subset E_N$ has finite measure, so $\mu(F_n) = \mu(E_N \setminus E_n) = \mu(E_N) - \mu(E_n)$. Thus $\lim \mu(F_n) = \mu(E_N) - \lim \mu(E_n)$. The equation $\mu(\bigcup_1^\infty F_n) = \lim \mu(F_n)$ is now $\mu(E_N) - \mu(\bigcap_1^\infty E_n) = \mu(E_N) - \lim \mu(E_n)$. Since $\mu(E_N) < \infty$, subtracting it from each side and transposing gives $\mu(\bigcap_1^\infty E_n) = \lim \mu(E_n)$.

(j) follows from (i) after defining $E_n = \bigcap_1^n E_j$.

(k) For each n , put $F_n = \bigcap_{j \geq n} E_j$. Then $\liminf E_n = \bigcup_{n=1}^\infty \bigcap_{j \geq n} E_j = \bigcup_1^\infty F_n$, and $F_1 \subset F_2 \subset \dots$. For $i \geq n$, $\mu(\bigcap_{j \geq n} E_j) \leq \mu(E_i)$ by monotonicity. Thus $\mu(\bigcap_{j \geq n} E_j) \leq \inf_{i \geq n} \mu(E_i)$. Continuity from below gives

$$\begin{aligned} \mu(\liminf E_n) &= \mu\left(\bigcup_1^\infty F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcap_{j \geq n} E_j\right) \leq \lim_{n \rightarrow \infty} \inf_{i \geq n} \mu(E_i) = \liminf \mu(E_n). \end{aligned}$$

(l) For each n put $F_n = \bigcup_{j \geq n} E_j$. Then $\limsup E_n = \bigcap_{n=1}^\infty \bigcup_{j \geq n} E_j = \bigcap_1^\infty F_n$, and $F_1 \supset F_2 \supset \dots$. Also, $F_N = \bigcup_N^\infty E_j$ has finite measure. For $i \geq n$, $\mu(\bigcup_{j \geq n} E_j) \geq \mu(E_i)$ by monotonicity. Thus $\mu(\bigcup_{j \geq n} E_j) \geq \sup_{i \geq n} \mu(E_i)$.

Continuity from above gives

$$\begin{aligned}\mu(\limsup E_n) &= \mu\left(\bigcap_1^\infty F_n\right) = \lim_{n \rightarrow \infty} \mu(F_n) \\ &= \lim_{n \rightarrow \infty} \mu\left(\bigcup_{j \geq n} E_j\right) \geq \lim_{n \rightarrow \infty} \sup_{i \geq n} \mu(E_i) = \limsup \mu(E_n).\end{aligned}$$

(m) It's clear that $\mu_E \geq 0$ and that $\mu_E(\emptyset) = 0$. If $\{E_n\}_1^\infty$ is disjoint, then

$$\mu_E\left(\bigsqcup_1^\infty E_n\right) = \mu\left(A \cap \bigsqcup_1^\infty E_n\right) = \mu\left(\bigsqcup_1^\infty A \cap E_n\right) = \sum_1^\infty \mu(A \cap E_n) = \sum_1^\infty \mu_E(E_n).$$

(n) Consider $\nu = \sum_1^n a_j \mu_j$, where μ_1, \dots, μ_n are measures on (X, \mathcal{M}) and $a_1, \dots, a_n \in [0, \infty)$. Since $a_j, \mu_j \geq 0$ for all j , the range of ν is contained in $[0, \infty]$. Clearly $\nu(\emptyset) = 0$. If $\{E_n\}_1^\infty$ are disjoint, then

$$\nu\left(\bigsqcup_{n=1}^\infty E_n\right) = \sum_{j=1}^n a_j \mu_j\left(\bigsqcup_{n=1}^\infty E_n\right) = \sum_{j=1}^n \sum_{n=1}^\infty a_j \mu_j(E_n) = \sum_{n=1}^\infty \sum_{j=1}^n a_j \mu_j(E_n) = \sum_{n=1}^\infty \nu(E_n),$$

the rearrangement valid because all terms of the double series are nonnegative. \square

Theorem 2. Let μ be a finitely additive measure on (X, \mathcal{M}) . Then

- (a) μ is a measure iff it is continuous from below.
- (b) μ is a measure iff it is continuous from above, provided $\mu(X) < \infty$.

Proof. (a) Assume μ is continuous from below. Let $\{E_n\}_1^\infty$ be disjoint and put $F_n = \bigsqcup_1^n E_j$. Then $\bigsqcup_1^\infty E_n = \bigcup_1^\infty F_n$, and $F_1 \subset F_2 \subset \dots$, and so

$$\mu\left(\bigsqcup_1^\infty E_n\right) = \mu\left(\bigcup_1^\infty F_n\right) = \lim \mu(F_n) = \lim \mu\left(\bigsqcup_1^n E_j\right) = \lim \sum_1^n \mu(E_j) = \sum_1^\infty \mu(E_n).$$

(The second equality is justified by continuity from below, and the fourth equality is justified by finite additivity.) The converse follows from Theorem 1.

(b) Assume μ is continuous from above and $\mu(X) < \infty$. Let $\{E_n\}_1^\infty$ be disjoint. Put $F_n = \bigsqcup_1^n E_j$. Then $\bigcup_1^\infty F_n = \bigsqcup_1^\infty E_n$, and $F_1^c \supset F_2^c \supset \dots$, and so

$$\begin{aligned}\mu\left(\bigsqcup_1^\infty E_n\right) &= \mu\left(\bigcup_1^\infty F_n\right) = \mu\left(X \setminus \bigcap_1^\infty F_n^c\right) = \mu(X) - \mu\left(\bigcap_1^\infty F_n^c\right) = \mu(X) - \lim \mu(F_n^c) \\ &= \mu(X) - \lim \mu(X \setminus F_n) = \mu(X) - \lim[\mu(X) - \mu(F_n)] = \lim \mu(F_n) \\ &= \lim \mu\left(\bigsqcup_1^n E_j\right) = \lim \sum_1^n \mu(E_j) = \sum_1^\infty \mu(E_n).\end{aligned}$$

The fourth equality is true by continuity from above, the third and sixth equalities because $\mu(X) < \infty$, and the ninth equality by finite additivity. \square

For technical reasons it is often easier to work with a complete measure. The next theorem shows that one may assume a measure is complete.

Theorem 3 (Completion of Measures). *Let (X, \mathcal{M}, μ) be a measure space. There is a unique smallest σ -algebra $\overline{\mathcal{M}}$ on X containing \mathcal{M} that includes all subsets of sets of measure zero, and a unique complete measure $\overline{\mu}$ defined on $\overline{\mathcal{M}}$ that extends μ (i.e. $\overline{\mu}(E) = \mu(E)$ for all $E \in \mathcal{M}$).*

Proof.

Step 1: (Construction of the σ -algebra $\overline{\mathcal{M}}$.) Define $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subset N \text{ for some } N \in \mathcal{N}\}$. Clearly $\overline{\mathcal{M}}$ is the unique smallest set containing \mathcal{M} and all subsets of μ -measure zero. It is now shown that $\overline{\mathcal{M}}$ is in fact a σ -algebra. If $\{E_n \cup F_n\}_1^\infty \subset \overline{\mathcal{M}}$, there exist sets $\{N_n\}_1^\infty \subset \mathcal{N}$ such that $F_n \subset N_n$ for each n . Put $E = \bigcup_1^\infty E_n$, $F = \bigcup_1^\infty F_n$, and $N = \bigcup_1^\infty N_n$. Then $E \in \mathcal{M}$, since \mathcal{M} is a σ -algebra, and $\mu(N) = 0$ by subadditivity, so $N \in \mathcal{N}$. Now $\bigcup_1^\infty E_n \cup F_n = (\bigcup_1^\infty E_n) \cup (\bigcup_1^\infty F_n) = E \cup F \in \overline{\mathcal{M}}$, since $E \in \mathcal{M}$ and $F \subset N \in \mathcal{N}$. Thus $\overline{\mathcal{M}}$ is closed under countable unions. It remains to show $\overline{\mathcal{M}}$ is closed under complementation. If $E \cup F \in \overline{\mathcal{M}}$, there exists an $N \in \mathcal{N}$ such that $F \subset N$. Then $E^c, N^c \in \mathcal{M}$, and $F^c = N^c \cup (N \setminus F)$. Thus

$$(E \cup F)^c = E^c \cap F^c = E^c \cap (N^c \cup (N \setminus F)) = (E^c \cap N^c) \cup (E^c \cap (N \setminus F)),$$

and this belongs to $\overline{\mathcal{M}}$ since $E^c \cap N^c \in \mathcal{M}$ and $E^c \cap (N \setminus F) \subset N \in \mathcal{N}$.

Step 2: (Construction of a complete measure $\overline{\mu}$ on $\overline{\mathcal{M}}$.) Define $\overline{\mu} : \overline{\mathcal{M}} \rightarrow [0, \infty]$ by $\overline{\mu}(E \cup F) = \mu(E)$. Suppose $E_1 \cup F_1$ and $E_2 \cup F_2$ belong to $\overline{\mathcal{M}}$ and $E_1 \cup F_1 = E_2 \cup F_2$. Then there are sets $N_1, N_2 \in \mathcal{N}$ such that $F_1 \subset N_1$ and $F_2 \subset N_2$. Then $E_1 \subset E_1 \cup F_1 = E_2 \cup F_2 \subset E_2 \cup N_2$, so $\overline{\mu}(E_1 \cup F_1) = \mu(E_1) \leq \mu(E_2 \cup N_2) = \mu(E_2) = \overline{\mu}(E_2 \cup F_2)$. Similarly, $\overline{\mu}(E_2 \cup F_2) \leq \overline{\mu}(E_1 \cup F_1)$, so in fact $\overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2)$. Thus $\overline{\mu}$ is well-defined. Clearly $\overline{\mu}(\emptyset) = 0$. If $\{E_n \cup F_n\}_1^\infty \subset \overline{\mathcal{M}}$ is a disjoint family, then the families $\{E_n\}_1^\infty \subset \mathcal{M}$ and $\{F_n\}_1^\infty \subset \mathcal{M}$ are disjoint as well, and

$$\overline{\mu}\left(\bigcup_1^\infty E_n \cup F_n\right) = \overline{\mu}\left(\left(\bigcup_1^\infty E_n\right) \cup \left(\bigcup_1^\infty F_n\right)\right) = \mu\left(\bigcup_1^\infty E_n\right) = \sum_1^\infty \mu(E_n) = \sum_1^\infty \overline{\mu}(E_n \cup F_n).$$

Therefore $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$. It is now shown to be complete. Suppose $E \cup F \in \overline{\mathcal{M}}$ has $\overline{\mu}(E \cup F) = \mu(E) = 0$. Then $E \in \mathcal{N}$, and there is an $N \in \mathcal{N}$ such that $F \subset N$. Thus $G \subset E \cup F \subset E \cup N$. Since $\mu(E) = \mu(N) = 0$, G is a subset of the μ -measure-zero set $E \cup N$; hence it belongs to $\overline{\mathcal{M}}$. Thus $\overline{\mu}$ is complete.

Step 3: (Uniqueness of $\overline{\mu}$.) Suppose ν is another measure on $\overline{\mathcal{M}}$ that extends μ , i.e. such that $\nu(E) = \mu(E)$ for all $E \in \mathcal{M}$. If $E \cup F \in \overline{\mathcal{M}}$, there is a set $N \in \mathcal{N}$ with $F \subset N$, and

$$\overline{\mu}(E \cup F) = \mu(E) = \nu(E) \leq \nu(E \cup F) \leq \nu(E \cup N) = \mu(E \cup N) = \mu(E) = \overline{\mu}(E \cup F).$$

(The equalities $\mu(E) = \nu(E)$ and $\nu(E \cup N) = \mu(E \cup N)$ are because $\nu = \mu$ on \mathcal{M} . The two inequalities are true by monotonicity of the measure ν .) The chain of inequalities shows that each must in fact be an equality, so $\overline{\mu}(E \cup F) = \nu(E \cup F)$ for any $E \cup F \in \overline{\mathcal{M}}$. This completes the proof of the theorem. \square

In Theorem 3, the σ -algebra $\overline{\mathcal{M}}$ is called the **completion** of \mathcal{M} and $\bar{\mu}$ is called the **completion** of μ .

Let (X, \mathcal{M}, μ) be a measure space. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, then μ is called **semifinite**. Given a measure μ on (X, \mathcal{M}) , define μ_0 on \mathcal{M} by $\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) < \infty\}$. Then μ_0 is called the **semifinite part** of μ . If the range of a measure is contained in $\{0, \infty\}$, then it is called **anti-semifinite**.

Theorem 4. *Every σ -finite measure is semifinite. If a measure is semifinite, then every subset of infinite measure contains subsets of arbitrarily large finite measure.*

The second conclusion says that if μ is semifinite and $\mu(E) = \infty$, then for any $C > 0$ there exists a measurable subset $F \subset E$ such that $C < \mu(F) < \infty$.

Proof.

Assume μ is a σ -finite measure on (X, \mathcal{M}) . Then there exists a sequence of measurable sets $\{F_j\}_1^\infty$ such that $X = \bigcup_1^\infty F_j$ and $\mu(F_j) < \infty$ for all j . Assume $\mu(E) = \infty$. Then $E = E \cap X = E \cap \bigcup_1^\infty F_j = \bigcup_1^\infty E \cap F_j$. Now, each $E \cap F_j$ is measurable and $\mu(E \cap F_j) \leq \mu(F_j) < \infty$. By subadditivity, $\mu(E) \leq \sum_1^\infty \mu(E \cap F_j)$. Since $\mu(E) = \infty$, it follows that $0 < \mu(E \cap F_N) < \infty$ for some (infinitely many) F_N . This proves the first part.

For the second part, let (X, \mathcal{M}, μ) be a semifinite measure space and assume $\mu(E) = \infty$. Put $\mathcal{F} = \{F \in \mathcal{M} : F \subset E, 0 < \mu(F) < \infty\}$, and define $r = \sup_{F \in \mathcal{F}} \mu(F)$. By semifiniteness, $\mathcal{F} \neq \emptyset$, and so $r \in (0, \infty]$. Choose a sequence $\{F_n\}_1^\infty \subset \mathcal{F}$ such that $\mu(F_n) \rightarrow r$ and let $G = E \setminus \bigcup_1^\infty F_n$. If $\mu(G) = \infty$, then semifiniteness provides a $G_0 \subset G$ with $G_0 \in \mathcal{F}$. But then $\mu(F_n \sqcup G_0) > r$ for sufficiently large n , a contradiction. Hence $0 \leq \mu(G) < \infty$. This implies $\infty = \mu(\bigcup_1^\infty F_n) \leq \sum_1^\infty \mu(F_n)$, and so $\mu(\bigcup_1^n F_j)$ can be made as desired by taking n sufficiently large. Since $\bigcup_1^n F_j \in \mathcal{F}$ for every n , the proof is complete. \square

A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets, called the **saturation** of \mathcal{M} . Clearly $\mathcal{M} \subset \tilde{\mathcal{M}}$. If $\mathcal{M} = \tilde{\mathcal{M}}$, then μ is called **saturated**. Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise. Then $\tilde{\mu}$ is called the **saturation** of μ .