Our goal is to construct a nontrivial measure on an arbitrary set. The plan is to first introduce the notion of an outer measure and show they are easy to come by. Next, Carathéodory's theorem, a powerful tool for constructing a complete measure from merely an outer measure, is proved. Finally, the notion of a premeasure on an algebra $\mathcal{A}$ is introduced, and it is shown that any premeasure on an algebra may be extended to a complete measure, with certain uniqueness properties, whose domain includes $\sigma(\mathcal{A})$.

Given a set X, an **outer measure** on X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that:

1. $\mu^*(\varnothing) = 0$,
2. $A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$ (**monotonicity**),
3. $\mu^*(\bigcup_1^\infty E_n) \leq \sum_1^\infty \mu^*(E_n)$ (**subadditivity**).

Theorem 1. *Let X be any set, $\{\varnothing, X\} \subset \mathcal{E} \subset \mathcal{P}(X)$, and $\mu_0 : \mathcal{E} \to [0, \infty]$ be such that $\mu_0(\varnothing) = 0$. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by*

$$\mu^*(E) = \inf \left\{ \sum_1^\infty \mu_0(E_n) : E_n \in \mathcal{E} \text{ and } E \subset \bigcup_1^\infty E_n \right\}.$$

Then μ^* is an outer measure on X, called the outer measure **induced** *from μ_0.*

Proof. The set over which the infimum is taken is nonempty because for any $E \in \mathcal{P}(X)$ one may take $E_1 = X$ and $E_n = \varnothing$ for $n \geq 2$. Thus μ^* is well-defined. $\mu(\varnothing) = 0$ since one may take $E_n = \varnothing$ for all n. Next, if $A \subset B$, the set over which the infimum is taken in the definition of $\mu^*(A)$ includes the set over which the infimum is taken in the definition of $\mu^*(B)$, so $\mu^*(A) \leq \mu^*(B)$. Finally, assume $\{E_n\}_1^\infty \subset \mathcal{P}(X)$. Let $\epsilon > 0$. By definition of μ^*, for each n there exists a sequence $\{E_n^k\}_{k=1}^\infty \subset \mathcal{E}$ such that $E_n \subset \bigcup_{k=1}^\infty E_n^k$ and $\sum_{k=1}^\infty \mu_0(E_n^k) \leq \mu^*(E_n) + \epsilon/2^n$. Summing, $\sum_{n,k=1}^\infty \mu_0(E_n^k) \leq \sum_{n=1}^\infty \mu^*(E_n) + \epsilon$. Thus

$$\mu^*(\bigcup_1^\infty E_n) = \inf\{\sum_1^\infty \cdots\} \leq \sum_{n,k=1}^\infty \mu_0(E_n^k) \leq \sum_{n=1}^\infty \mu^*(E_n) + \epsilon.$$

(The first inequality is true because $\bigcup_1^\infty E_n \subset \bigcup_{n,k=1}^\infty E_n^k$, so $\sum_{n,k} \mu_0(E_n^k)$ belongs to the set over which the infimum is taken.) Since ϵ is arbitrary, we see that $\mu^*(\bigcup_1^\infty E_n) \leq \sum_1^\infty \mu^*(E_n)$. □

Thus an outer measure μ^* can be constructed on *every* set X: just choose $\mathcal{E} = \{\varnothing, X\}$, define $\mu_0(\varnothing) = 0$, $\mu_0(X) = 1$, and let μ^* be the outer measure induced by μ_0. (Admittedly, this is not a very interesting outer measure.) Let μ^* be an outer measure on X. A set $A \subset X$ is called **μ^*-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for every } E \subset X.$$

That is, A is μ^*-measurable if it "splits" the outer measure of every set $E \subset X$ into two additive parts: $\mu^*(E \cap A)$ and $\mu^*(E \cap A^c)$. Subadditivity of outer measures implies $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The reverse inquality $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is obvious if $\mu^*(E) = \infty$. Thus:

Lemma 2. *The set A is μ^*-measurable iff $\mu^*(E) < \infty$ implies*

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

The principal tool of this section, Carathéodory's theorem, may now be presented. This theorem gives us a tool for constructing a complete measure from merely an outer measure.

Theorem 3 (Carathéodory's Theorem). *Let μ^* be an outer measure, and let $\mathcal{M}$ be the collection of μ^*-measurable sets. Then:*

1. *$\mathcal{M}$ is a σ-algebra.*
2. *$\mu := \mu^*|_{\mathcal{M}}$ is a complete measure on $\mathcal{M}$.*

Proof.

Step 1: This step shows that $\mathcal{M}$ is an algebra and that μ^* is finitely additive on $\mathcal{M}$. By the definition of μ^*-meaurable, $\mathcal{M}$ is closed under taking complements. If $A, B \in \mathcal{M}$ and $E \subset X$, then

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

Since $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cup B)$, subadditivity of μ^* gives

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B)).$$

Because $A^c \cap B^c = (A \cup B)^c$, it follows that

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Consequently, $A \cup B \in \mathcal{M}$ whenever A and B are. So $\mathcal{M}$ is an algebra. Next, if $A \cap B = \varnothing$ then

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B),$$

the equality being true because $(A \cup B) \cap A^c = B \cap A^c = B \setminus A = B$, since $A \cap B = \varnothing$. Thus μ^* is finitely additive on $\mathcal{M}$.

Step 2: This step shows that $\mathcal{M}$ is a σ-algebra and that μ^* is a measure on $\mathcal{M}$. Assume $\{A_j\}_1^\infty \subset \mathcal{M}$ is a disjoint sequence. Put $B_n = \bigcup_1^n A_j$ and $B = \bigcup_1^\infty A_j$. Let $E \subset X$. By Step 1, $\mathcal{M}$ is an algebra, so each A_n belongs to $\mathcal{M}$ (i.e. each A_n is μ^*-measurable), and so

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}),\end{aligned}$$

the equality being true because $A_n \subset B_n$ and because $B_n \cap A_n^c = (\bigcup_1^n A_j) \setminus A_n = \bigcup_1^{n-1} A_j = B_{j-1}$. By induction, $\mu^*(E \cap B_n) = \sum_1^n \mu^*(E \cap A_j)$, so

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &\geq \mu^*(E \cap B_n) + \mu^*(E \cap B^c) = \sum_1^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c),\end{aligned}$$

the inequality being true because $B_n^c \supset B^c$. As $n \to \infty$, subadditivity gives

$$\begin{aligned}\mu^*(E) &\geq \sum_1^\infty \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*(\bigcup_1^\infty (E \cap A_j)) + \mu^*(E \cap B^c) \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E).\end{aligned}$$

The preceding display had the form $\mu^*(E) \geq \cdots \geq \mu^*(E)$, so all inequalities are in fact equalities. Thus $B \in \mathcal{M}$. Since $E \subset X$ was arbitrary, one may take $E = B$ in the preceding string of equalities to obtain

$$\mu^*(B) = \mu^*(\bigcup_1^\infty A_j) = \sum_1^\infty \mu^*(B \cap A_j) = \sum_1^\infty \mu^*(A_j).$$

Since $\mathcal{M}$ is closed under taking complements, and $\mu^*(\bigcup_1^\infty A_j) = \sum_1^\infty \mu^*(A_j)$ whenever $\{A_j\}$ is a disjoint sequence in $\mathcal{M}$, μ^* is a measure on $\mathcal{M}$.

Step 3: Finally, it is shown that μ is complete. Suppose $A \in \mathcal{M}$ has $\mu^*(A) = 0$, and let $B \subset A$. It must be established that $B \in \mathcal{M}$. For $E \subset X$,

$$\mu^*(E) \leq \mu^*(E \cap B) + \mu^*(E \cap B^c) = \mu^*(E \cap B^c) \leq \mu^*(E),$$

where the first inequality followed by subadditivity, and the rest followed by monotonicity. The preceding inequalities are therefore all equalities, so that $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c)$ for any $E \subset X$. This shows that $B \in \mathcal{M}$. □

Existence of μ^*-measurable sets (other than $\varnothing$ and X) is not readily apparent. Premeasures and the upcoming theorem remedy this complaint. Let $\mathcal{A}$ be an algebra on X. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is a **premeasure** if:

1. $\mu_0(\varnothing) = 0$,
2. $\mu_0(\bigcup_1^\infty A_n) = \sum_1^\infty \mu_0(A_n)$ whenever $\{A_n\} \subset \mathcal{A}$ is a disjoint family and $\bigcup_1^\infty A_n \in \mathcal{A}$.

A key distinction between premeasures and measures is that a premeasure is defined on an algebra, not necessarily a σ-algebra. Thus $\{A_n\} \subset \mathcal{A}$ need not imply $\bigcup_1^\infty A_n \in \mathcal{A}$. This explains the reason for the "and" conjunction in part 2 of the definition. If $\mathcal{A}$ is actually a σ-algebra, then premeasures and measures on $\mathcal{A}$ coincide. Finally, any premeasure on $\mathcal{A}$ satisfies the hypotheses of Theorem 1, so one may speak of the outer measure induced by a premeasure.

Theorem 4. *Let μ_0 be a premeasure on an algebra $\mathcal{A}$ of subsets of X, and let μ^* be the induced outer measure. Then:*

1. *$\mu^*|_{\mathcal{A}} = \mu_0$ (that is, μ^* extends μ_0),*
2. *$\sigma(\mathcal{A}) \subset \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra of μ^*-measurable sets.*

Proof.

To establish (1), it is shown that $\mu^*(E) = \mu_0(E)$ for all $E \in \mathcal{A}$. Indeed for any such E, let $\{A_j\}_1^\infty \subset \mathcal{A}$ be such that $E \subset \bigcup_1^\infty A_j$. The inequality $\mu^*(E) \leq \mu_0(E)$ follows by taking $A_1 = E$ and $A_n = \varnothing$ for $n \geq 2$ in the definition of the induced outer measure. For the reverse inequality, put $B_n = E \cap (A_n \setminus \bigcup_1^{n-1} A_j)$. Then $\{B_n\}_1^\infty$ is a disjoint family of members of $\mathcal{A}$ whose union is E. Therefore,

$$\mu_0(E) = \mu_0(\bigcup_1^\infty B_j) = \sum_1^\infty \mu_0(B_j) \leq \sum_1^\infty \mu_0(A_j),$$

where the second equality followed by the definition of a premeasure. Since the choice of the covering $\{A_j\}_1^\infty$ of E was arbitrary, $\mu_0(E) \leq \mu^*(E)$ from the display and the definition of the induced outer measure.

For part (2), Let $A \in \mathcal{A}$, $E \subset X$, and $\epsilon > 0$. Then there exists a sequence $\{B_j\}_1^\infty \subset \mathcal{A}$ such that $E \subset \bigcup_1^\infty B_j$ and $\sum_1^\infty \mu_0(B_j) \leq \mu^*(E) + \epsilon$. By additivity of μ_0 on $\mathcal{A}$,

$$\mu^*(E) + \epsilon \geq \sum_1^\infty \mu_0(B_j \cap A) + \sum_1^\infty \mu_0(B_j \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since ϵ was arbitrary, we have $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Since $E \subset X$ was arbitrary, this shows that A is μ^*-measurable. □

The following theorem gathers together the results of this section into a form frequently desirable in applications.

Theorem 5. *Let μ_0 be a premeasure on an algebra $\mathcal{A}$. Then:*

1. *There exists a complete measure μ on a σ-algebra $\mathcal{M} \supset \sigma(\mathcal{A})$ such that μ extends μ_0 (i.e. $\mu|_{\mathcal{A}} = \mu_0$).*
2. *If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$.*
3. *If μ_0 is σ-finite, then μ is σ-finite and is the unique extension of μ_0 to a measure on $\mathcal{M}$.*

Proof.

For the first part, let μ^* be the outer measure induced by μ_0. By Carathéodory's theorem, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and $\mu = \mu^*|_{\mathcal{M}}$ is a complete measure on $\mathcal{M}$. By Theorem 4, $\mathcal{M} \supset \sigma(\mathcal{A})$ and μ extends μ_0.

For the second part, let $E \in \mathcal{M}$ and choose $\{A_n\} \subset \mathcal{A}$ such that $E \subset \bigcup_1^\infty A_j$. Now, $\nu(E) \leq \sum_1^\infty \nu(A_j) = \sum_1^\infty \mu_0(A_j) \leq \mu(E)$. (The first inequality is true because ν is a measure, the equality is true because $\nu = \mu_0$ on $\mathcal{A}$, and the final inequality is true because $\mu = \mu^*|_{\mathcal{M}}$ and from the definition of the induced outer measure.) If $\mu(E) < \infty$, it is possible to choose $\{A_j\}$ such that $\mu(A) < \mu(E)+\epsilon$, where $A = \bigcup_1^\infty A_j$. Then $\nu(A) = \lim \nu(\bigcup_1^n A_j) = \lim \mu(\bigcup_1^n A_j) = \mu(A)$. (The first and third equalities are true by continuity from below of measures, and the middle equality is true because $\nu = \mu_0 = \mu$ on $\mathcal{A}$.) Since $\mu(E) < \infty$, it follows that $\mu(A \setminus E) < \epsilon$, and

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) < \nu(E) + \epsilon.$$

Thus $\nu(E) \leq \mu(E) < \nu(E) + \epsilon$. Since ϵ is arbitrary, $\mu(E) = \nu(E)$.

For the last part, if μ_0 is σ-finite, choose $\{A_j\} \subset \mathcal{A}$ such that $X = \bigcup_1^\infty A_j$ and $\mu_0(A_j) < \infty$ for all j. Since $\mu = \mu_0$ on $\mathcal{A}$, it follows that μ is σ-finite as well. Furthermore, one may assume the family $\{A_j\}$ is disjoint (by replacing A_n with $A_n \setminus \bigcup_1^{n-1} A_j$ if necessary.) Now let $E \in \mathcal{M}$. Then $E = E \cap X = E \cap \bigsqcup_1^\infty A_j$, and since $\mu = \nu$ on $\mathcal{M}$,

$$\mu(E) = \mu(E \cap \bigsqcup_1^\infty A_j) = \sum_1^\infty \mu(E \cap A_j) = \sum_1^\infty \nu(E \cap A_j) = \nu(E \cap \bigsqcup_1^\infty A_j) = \nu(E).$$

Thus $\mu = \nu$, so μ is unique. □

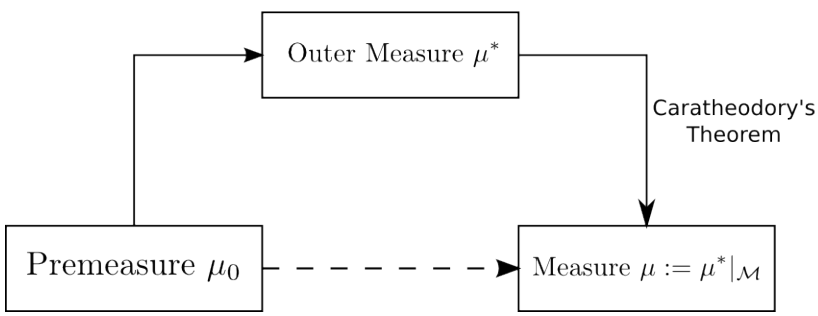

Figure 1: Construction of Measures