

If \mathcal{E} is any subset of $\mathcal{P}(X)$, then the intersection of all σ -algebras on X that contain \mathcal{E} is the unique smallest σ -algebra on X containing \mathcal{E} . It is called the **σ -algebra generated by \mathcal{E}** and denoted $\sigma(\mathcal{E})$. If X is any topological space, the **Borel σ -algebra** on X , denoted \mathcal{B}_X , is the σ -algebra generated by the family of open sets in X . That is, if τ is the topology on X , then $\mathcal{B}_X = \sigma(\tau)$. If \mathcal{M}_α is a σ -algebra on X_α for each α in an index set A , then the **product σ -algebra** on $X = \prod_{\alpha \in A} X_\alpha$, denoted $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$, is the σ -algebra generated by the collection

$$\mathcal{E} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathcal{M}_\alpha\},$$

where $\pi_\alpha : X \rightarrow X_\alpha$ is the canonical projection map.

Theorem 1.

1. If \mathcal{M} is any σ -algebra that contains \mathcal{E} , then $\sigma(\mathcal{E}) \subset \mathcal{M}$. Thus if $\mathcal{E} \subset \sigma(\mathcal{E}')$, then $\sigma(\mathcal{E}) \subset \sigma(\mathcal{E}')$.
2. If \mathcal{M}_α is generated by \mathcal{E}_α for each $\alpha \in A$, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by

$$\{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathcal{E}_\alpha\}.$$

3. If X is a second countable topological space, then \mathcal{B}_X is generated by any countable base for X .
4. If $\{X_\alpha\}_{\alpha \in A}$ is any collection of topological spaces and $X = \prod_{\alpha \in A} X_\alpha$, then

$$\bigotimes_{\alpha \in A} \mathcal{B}_{X_\alpha} \subset \mathcal{B}_X.$$

In words, the product Borel σ -algebra is contained in the Borel σ -algebra of the product space.

5. If $\{X_n\}_1^\infty$ is any countable collection of second countable topological spaces and $X = \prod_1^\infty X_n$, then

$$\bigotimes_1^\infty \mathcal{B}_{X_n} = \mathcal{B}_X.$$

Proof.

1. Let \mathcal{F} be the family of all σ -algebras on X that contain \mathcal{E} . Then $\sigma(\mathcal{E}) = \bigcap_{F \in \mathcal{F}} F \subset \mathcal{M}$, since $\mathcal{M} \in \mathcal{F}$.
2. Put $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathcal{E}_\alpha\}$ and $\mathcal{E} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \mathcal{M}_\alpha\}$. Since $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha = \sigma(\mathcal{E})$, it suffices to show that $\sigma(\mathcal{F}) = \sigma(\mathcal{E})$. Since $\mathcal{F} \subset \mathcal{E}$, clearly $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$. On the other hand, for any $\alpha \in A$, the collection $\{E \subset X_\alpha : \pi_\alpha^{-1}(E) \in \sigma(\mathcal{F})\}$ is a σ -algebra that contains \mathcal{E}_α , and therefore contains \mathcal{M}_α , by part 1. Thus for any $\alpha \in A$ and $E_\alpha \in \mathcal{M}_\alpha$, $\pi_\alpha^{-1}(E_\alpha) \in \sigma(\mathcal{F})$. Consequently, $\mathcal{E} \subset \sigma(\mathcal{F})$, so $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$.

3. Let \mathcal{E} be a countable base for the topology τ on X . Clearly $\mathcal{E} \subset \tau$, so $\sigma(\mathcal{E}) \subset \sigma(\tau) = \mathcal{B}_X$. On the other hand, any $U \in \tau$ is a countable union of members of \mathcal{E} . Thus $U \in \sigma(\mathcal{E})$ for all $U \in \tau$. Consequently, $\tau \subset \sigma(\mathcal{E})$, and so $\mathcal{B}_X = \sigma(\tau) \subset \sigma(\mathcal{E})$.
4. Let τ_α denote the topology on X_α and τ denote the product topology on X . By definition of the product topology, $\pi_\alpha^{-1}(E_\alpha)$ is open in X for each $\alpha \in A$ and E_α in τ_α . Therefore,

$$\mathcal{E} = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in \tau_\alpha\} \subset \tau.$$

Consequently,

$$\bigotimes_{\alpha \in A} \mathcal{B}_{X_\alpha} = \sigma(\mathcal{E}) \subset \sigma(\tau) = \mathcal{B}_X.$$

5. By part 4 we have $\bigotimes_1^\infty \mathcal{B}_{X_n} \subset \mathcal{B}_X$. We now obtain the reverse inclusion. Being second countable, each X_n has a countable base \mathcal{B}_n . The collection \mathcal{F} of all products $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i , is a countable base for X . Thus to each product $\prod U_i$ in \mathcal{F} , there correspond positive integers j_1, j_2, \dots, j_k such that

$$\prod U_i = \pi_{j_1}^{-1}(U_{j_1}) \cap \dots \cap \pi_{j_k}^{-1}(U_{j_k}).$$

Each $\pi_{j_i}^{-1}(U_{j_i})$ belongs to

$$\mathcal{E} = \{\pi_n^{-1}(E_n) : n \in \mathbb{N}, E_n \in \mathcal{B}_n\},$$

and since $\bigotimes_1^\infty \mathcal{B}_{X_n} = \sigma(\mathcal{E})$ is a σ -algebra that contains each $\pi_{j_i}^{-1}(U_{j_i})$, it contains the intersection $\prod U_i = \pi_{j_1}^{-1}(U_{j_1}) \cap \dots \cap \pi_{j_k}^{-1}(U_{j_k})$ as well. That is, $\prod U_i \in \sigma(\mathcal{E})$. This for all $\prod U_i \in \mathcal{F}$, so $\mathcal{F} \subset \sigma(\mathcal{E})$. By part 3, $\mathcal{B}_X = \sigma(\mathcal{F})$. Applying part 1 now gives

$$\mathcal{B}_X = \sigma(\mathcal{F}) \subset \sigma(\mathcal{E}) = \bigotimes_1^\infty \mathcal{B}_{X_n}.$$

□

Corollary 2. *For any positive integer n , $\bigotimes_1^n \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^n}$. Furthermore, $\bigotimes_1^\infty \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^\omega}$. Also, since the topology on \mathbb{C} and \mathbb{R}^2 are identical, $\mathcal{B}_{\mathbb{C}} = \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.*