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Most texts on real analysis give a terse, opaque proof that a measurable function is the limit of simple functions. This statement is so important that we motivate the key ideas and give an expanded proof.

Theorem 1. *If $f : X \to [0,\infty]$ is measurable, then there exist real-valued simple functions $s_1, s_2, \ldots$ on X such that $0 \leq s_1 \leq s_2 \leq \cdots \leq f$, $s_n(x) \to f(x)$ pointwise (and uniformly on any set for which f is bounded).*

Proof. The basic idea is to construct a sequence of simple functions $\phi_1, \phi_2, \ldots$ that approximate the identity from below, for then $s_n = \phi_n \circ f$ is a sequence of simple functions that approximate f from below.

Step 1 (Approximation of the Identity): Define $\phi_n : [0,\infty] \to [0,\infty)$ by

$$\phi_n(t) = \begin{cases} \lfloor 2^n t \rfloor / 2^n & \text{if } t \in [0,n) \\ n & \text{if } t \in [n,\infty] \end{cases},$$

where $\lfloor \cdot \rfloor$ denotes the least integer function.[1]

Each ϕ_n is Borel measurable since for any $\alpha \in \mathbb{R}$, $\phi_n^{-1}((\alpha,\infty])$ is an interval of the form $[t,\infty]$, a Borel set. First we show $\phi_n \leq \phi_{n+1}$. For $t \in [0,\infty)$, consider the three cases: (i) $t \in [0,n)$, (ii) $t \in [n, n+1)$, and (iii) $t \in [n+1,\infty]$. In case (i), $\phi_n(t) = \frac{\lfloor 2^n t \rfloor}{2^n} = \frac{2 \cdot \lfloor 2^n t \rfloor}{2 \cdot 2^n} \leq \frac{\lfloor 2^{n+1} t \rfloor}{2^{n+1}} = \phi_{n+1}(t)$, where the inequality follows because $n\lfloor t \rfloor \leq \lfloor nt \rfloor$. In case (ii), $\phi_n(t) = n \leq \lfloor t \rfloor = \frac{2^{n+1}\lfloor t \rfloor}{2^{n+1}} \leq \frac{\lfloor 2^{n+1} t \rfloor}{2^{n+1}} = \phi_{n+1}(t)$. In case (iii), $\phi_n(t) = n < n+1 = \phi_{n+1}(t)$. If $t = \infty$, then $\phi_n(t) = n < n+1 = \phi_{n+1}(t)$. Thus for all $t \in [0,\infty]$ and all n, $\phi_n(t) \leq \phi_{n+1}(t) \leq t$. Consequently, $0 \leq \phi_1(t) \leq \phi_2(t) \leq \cdots \leq t$ for all $t \in [0,\infty]$.

Now we show that $\phi_n(t) \to t$ for $t \in [0,\infty]$. If $t \in [0,\infty)$, then $2^n t - 1 < \lfloor 2^n t \rfloor$ and so $\frac{2^n t - 1}{2^n} < \frac{\lfloor 2^n t \rfloor}{2^n}$ which implies $t - \frac{1}{2^n} < \frac{\lfloor 2^n t \rfloor}{2^n}$. As soon as $n > t$, $\phi_n(t) = \lfloor 2^n t \rfloor / 2^n$, so $t - \frac{1}{2^n} < \phi_n(t) \leq t$ for $n > t$. Hence $|\phi_n(t) - t| < \frac{1}{2^n}$ for $n > t \in [0,\infty)$, so $\phi_n(t) \to t$. If $t = \infty$ then $\phi_n(t) = n$ for all n, so $\phi_n(t) \to \infty = t$. So, $\phi_n(t) \to t$ for all $t \in [0,\infty]$ as claimed.

Step 2 (Approximation of f): Composing any function on the left with a simple function gives another simple function. Thus each $s_n : X \to [0,\infty)$ defined by $s_n = \phi_n \circ f$ is simple. Also, each s_n is measurable since f is measurable and ϕ_n is Borel measurable.

For any n and $x \in X$, $0 \leq s_n(x) = \phi_n(f(x)) \leq \phi_{n+1}(f(x)) = s_{n+1}(x)$, so $0 \leq s_1 \leq s_2 \leq \cdots$. Moreover, $s_n(x) = \phi_n(f(x)) \leq f(x)$ for every n and x, so in fact $0 \leq s_1 \leq s_2 \leq \cdots \leq f$, which proves the first assertion of the theorem.

To prove convergence, let $x \in X$. Then $f(x) \in [0,\infty]$, and so

$$\lim s_n(x) = \lim \phi_n(f(x)) = f(x),$$

because $\phi_n(t) \to t$ for all $t \in [0,\infty]$. That is, $s_n(x) \to f(x)$ on X. In fact, since $|s_n(x) - f(x)| = |\phi_n(f(x)) - f(x)| < \frac{1}{2^n}$ whenever $f(x) \in [0,\infty)$, the convergence is uniform on any set on which f is bounded. $\square$

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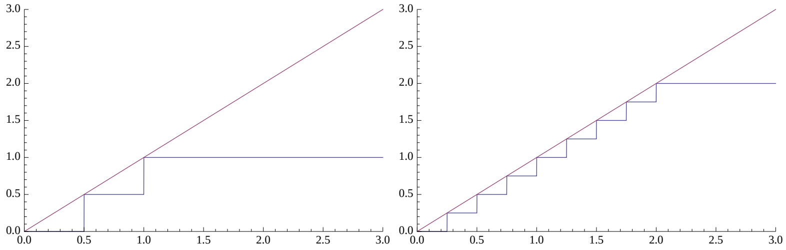
[1] Equivalently, Rudin defines $k_n(t)$ to be the unique positive integer such that $k2^{-n} \leq t < (k+1)2^{-n}$, and then $\phi_n(t) = k_n(t)2^{-n}$ if $t \in [0,n)$ and $\phi_n(t) = n$ if $t \in [n,\infty]$.

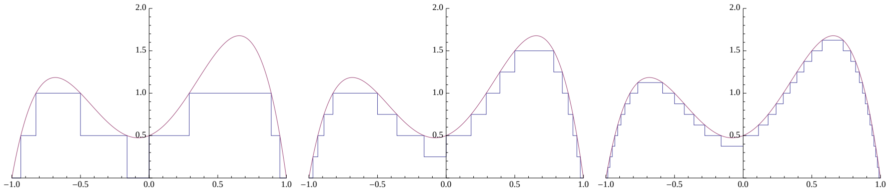


(a) $\phi_1(t)$

(b) $\phi_2(t)$

Figure 1: The approximations $\phi_1(t), \phi_2(t)$ to the identity function.

Figure 2 shows that graphs (in blue) of the approximations ϕ_1, ϕ_2, and ϕ_3 to the function $f(x)$ (in magenta) on $X = [-1, 1]$.



(a) ϕ_1

(b) ϕ_2

(c) ϕ_3

Figure 2: The simple approximations ϕ_1, ϕ_2, ϕ_3 to $f(x)$ over $X = [-1, 1]$.